# Estimation and inference for higher-order stochastic volatility models with leverage

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# Objective

Propose simple methods for estimation and inference of higher-order stochastic volatility models with leverage, SVL($p$)

- Time-varying/dynamic volatility
- Underlying volatility process follows an AR($p$) process
- Leverage effect: inverse relationship between observed process and its volatility
  - Can Improve model fit
  - Empirically relevant for financial data
  - Improves forecast performance

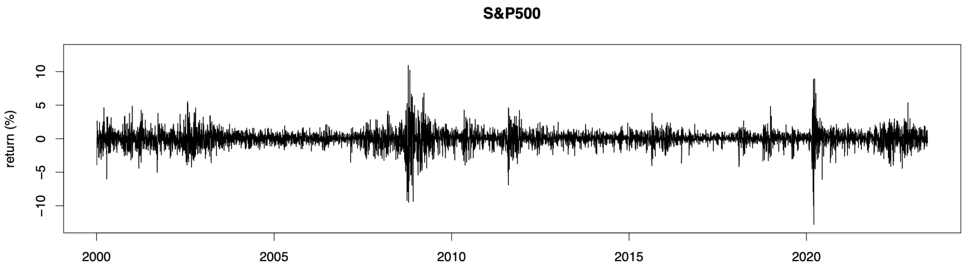
# Introduction

## Why do we care about time-varying volatility?

- Finance: Dynamic volatility has consequences for many problems of financial decision
  - Risk management
  - Portfolio allocation
  - Asset pricing
- Macroeconomics: Dynamic volatility is also important for macroeconomic forecasting, measurement of uncertainty, and identification of structural shocks
  - See Cogley and Sargent (2005), Primiceri (2005), Benati (2008),Koop et al. (2009), Koop and Korobilis (2013), Liu and Morley (2014), Jurado et al. (2015)

# Example

Figure 1: TS of S&P500 daily returns from Jan-2000 to May-2023 ($T = 5,889$)




Stylized facts in financial data:

- Volatility clustering
- Persistent volatility process
- Leverage effect: tendency of an asset's volatility to be negatively correlated with the asset's returns

# Motivation

Two main classes of models have been proposed for dynamic volatility

1. GARCH-type models [Engle (1982)]
   - Volatility is modelled as a deterministic process
   - Example: GARCH(1,1)

$$
\begin{aligned}
y_t &= \sigma_t z_t, \\
\sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta \sigma_{t-1}^2
\end{aligned}
$$

2. Stochastic Volatility (SV) models [Taylor (1982),Taylor (1986)]
   - Volatility is modelled as a latent stochastic process
   - Example: SV(1)

$$
\begin{aligned}
y_t &= \sigma_t z_t, \\
\log(\sigma_t^2) &= \alpha + \phi_1 \log(\sigma_{t-1}^2) + \nu_t
\end{aligned}
$$

# Motivation

SV models may be preferable to GARCH-type models for several reasons.

- Discrete version of continuous time analogues [Shephard and Andersen (2009), Taylor (1994)]
- Empirical evidence suggests SV models are robust to model misspecification [Carnero et al. (2004), Chan and Grant (2016)]
- Provide more accurate forecasts of volatility [Kim et al. (1998), Yu (2002), Poon and Granger (2003), Koopman et al. (2005)]
- Statistical properties of SV models are relatively easy to derive [Davis and Mikosch (2009)]

Empirical popularity of SV models are deterred by two reasons:

1. No closed form solution for the likelihood function
2. Limited statistical packages for SV models vs. many for GARCH

# Motivation

What are the alternative SV estimator?

- Proposed estimation methods of SV models are limited to SV(1):
  - Generalized Method of Moments
  - Simulated Maximum likelihood (SML)
  - Quasi Maximum Likelihood (QML)
  - Bayesian techniques based on Markov Chain Monte Carlo (MCMC)
  - Simulated Method of Moments (SMM)
  - Monte Carlo likelihood (MCL)
  - Linear-representation based estimation (LR)
  - Moment based closed-form estimator (DV)
  - ARMA-SV estimator (ARMA-SV)

- With exception of ARMA-SV, these estimation methods are either inefficient and/or very expensive from the computational viewpoint, inflexible across models, not easy to implement in practice, and may not converge [Broto and Ruiz (2004), Ahsan and Dufour (2019)].

# Contributions

In this paper, we extend Ahsan and Dufour (2021) by considering estimation of higher order SV models with leverage

Contributions:

- Estimation
  - ★ Develop a closed-form estimator for SVL($p$)
  - Utilizes associated ARMA representation of SVL models
  - Computationally efficient
- Inference
  - Monte Carlo tests for testing hypothesis of no leverage (i.e., $H_0 : \delta = 0$)
  - ★ Exact finite-sample test procedure
- Empirical application
  - Find evidence of significant leverage effect in daily returns
  - ★ Forecasting volatility with SVL($p$) outperforms competing models: ARCH, GARCH, EGARCH, GJR, SV($p$)

# SVL($p$) model

Consider a discrete-time SV process of order ($p$) with leverage (SVL($p$))

$$y_t = \sigma_y \exp(\frac{w_t}{2}) z_t \,, \quad z_t \sim \text{i.i.d. } \mathcal{N}(0,\, 1)\,, \tag{1}$$

$$w_t = \sum_{j=1}^{p} \phi_j w_{t-j} + \sigma_v v_t \,, \quad v_t \sim \text{i.i.d. } \mathcal{N}(0,\, 1)\,, \tag{2}$$

$$\mathbb{E}[z_{t-1}, v_t] = \mathrm{corr}(z_{t-1}, v_t) = \delta \tag{3}$$

and by normality of $z_t$ and $v_t$,

$$v_t = \delta z_{t-1} + \sqrt{1-\delta^2}\,\tilde{v}_t \tag{4}$$

where $\tilde{v}_t \sim N(0,1)$

# SVL($p$) model: State-space form

Which can be written in state-space form as:

$$y_t^* = w_t + \epsilon_t \,, \tag{5}$$

$$w_t = \sum_{j=1}^{p} \phi_j w_{t-j} + \delta \sigma_\nu z_{t-1} + \sqrt{1-\delta^2}\sigma_\nu \tilde{v}_t \,, \tag{6}$$

where

$$y_t^* = \log(y_t^2) - \mu, \qquad \mu = \log(\sigma_y^2) + \mathbb{E}[\log(z_t^2)], \qquad \epsilon_t = \log(z_t^2) - \mathbb{E}[\log(z_t^2)]$$

with normality of $z_t$, $\epsilon_t$ follows a centered $i.i.d.$ $\log\chi^2$ and so

$$\mathbb{E}[\log(z_t^2)] \simeq -1.2704, \qquad \sigma_\epsilon := \mathbb{E}[\epsilon_t^2] = \frac{\pi^2}{2}, \qquad \mathbb{E}[\epsilon_t^3] = \psi^{(2)}(\frac{1}{2}), \qquad \mathbb{E}[\epsilon_t^4] = \pi^4 + 3\sigma_\epsilon^2$$

where $\psi^{(2)}(z)$ is the polygamma function of order 2.

# SVL($p$) model: ARMA representation

This model the following ARMA($p, p$) representation:

$$y_t^* = \sum_{j=1}^{p} \phi_j y_{t-j}^* + \eta_t - \sum_{j=1}^{p} \theta_j \eta_{t-j} \,, \tag{7}$$

$$\eta_t - \sum_{j=1}^{p} \theta_j \eta_{t-j} = v_t + \epsilon_t - \sum_{j=1}^{p} \phi_j \epsilon_{t-j}, \tag{8}$$

where $\{v_t\}$ and $\{\epsilon_t\}$ are mutually independent error processes, the errors $v_t$ are i.i.d. $\mathcal{N}(0, \sigma_v^2)$, and the errors $\epsilon_t$ are i.i.d. according to the distribution of a $\log(\chi_1^2)$ random variable.

# Estimation: Moments

The auto-covariances of the observed process $y_t^*$ are given by

$$\operatorname{cov}(y_t^*, y_{t-k}^*) := \gamma_{y^*}(k) = \begin{cases} \phi_1\gamma_{y^*}(k-1) + \cdots + \phi_p\gamma_{y^*}(k-p) + \sigma_v^2 + \sigma_\epsilon^2 & \text{if } k = 0 \\ \phi_1\gamma_{y^*}(k-1) + \cdots + \phi_p\gamma_{y^*}(k-p) - \phi_k\sigma_\epsilon^2 & \text{if } 1 \leq k \leq p \\ \phi_1\gamma_{y^*}(k-1) + \cdots + \phi_p\gamma_{y^*}(k-p) & \text{if } k > p \end{cases} \tag{9}$$

Moment equation involving unsquared leverage parameter

$$\mathbb{E}\left[|y_t|y_{t-1}\right] = \frac{\delta\sigma_v\sigma_y^2}{\sqrt{2\pi}} \exp\left(\tilde{\gamma}/4\right) \tag{10}$$

where $\tilde{\gamma} := \operatorname{var}(w_t) + \operatorname{cov}(w_t, w_{t-1})$.

# Estimation: ARMA-based estimatiors

Closed-form expressions for SVL(p) parameters are given by:

$$\boldsymbol{\phi}_p = \boldsymbol{\Gamma}(p, j)^{-1}\boldsymbol{\gamma}(p, j), \quad j \geq 1\,, \tag{11}$$

$$\sigma_y = [\exp(\mu + 1.2704)]^{1/2}\,, \tag{12}$$

$$\sigma_v = [\gamma_{y^*}(0) - \boldsymbol{\phi}_p^{'}\boldsymbol{\gamma}(1) - \pi^2/2]^{1/2}, \tag{13}$$

$$\delta = \frac{\sqrt{2\pi}\lambda_y(1)}{\sigma_v\sigma_y^2}\exp\Big(-\frac{1}{4}\tilde{\gamma}\Big)\,, \tag{14}$$

where

- $\boldsymbol{\Gamma}(p,j)$ is a $p \times p$ matrix of autocovariance
- $\boldsymbol{\gamma}(p, j) := [\gamma_{y^*}(p+j), \ldots, \gamma_{y^*}(2p+j-1)]^{'}$
- $\boldsymbol{\gamma}(p) := [\gamma_{y^*}(1), \ldots, \gamma_{y^*}(p)]^{'}$
- $\boldsymbol{\phi}_p := (\phi_1, \ldots, \phi_p)^{'}$
- $\gamma_{y^*}(k) = \mathrm{cov}(y_t^*, y_{t-k}^*)$
- $\lambda_y(1) := \mathbb{E}[|y_t|y_{t-1}]$

# Estimation: Winsorized estimation

Can achieve better stability and efficiency of CF-ARMA estimator by using "winsorization". From (11), we can use

$$\phi_p = \sum_{j=1}^{\infty} \omega_j \mathbf{\Gamma}(p, j)^{-1} \gamma(p, j) \tag{15}$$

where $\sum_{j=1}^{J} \omega_j = 1$, $1 \leq J \leq T - p$, and $T$ is the length of time series.

Paper includes a full list of W-ARMA methods (see also Ahsan and Dufour (2021) within the framework of SV($p$) models)

# Estimation: OLS-based W-ARMA estimation

In this paper, we use the OLS-based W-ARMA estimator based on an OLS regression without intercept:

$$\hat{\phi}_p^{\text{ols}} = [A(p, J)'A(p, J)]^{-1}A(p, J)'e(p, J) \tag{16}$$

where $e(p, J)$ is a $(pJ) \times 1$ vector and $A(p, J)$ is a $(pJ) \times p$ matrix defined by

$$e(p, J) = [\hat{\gamma}(p, 1)\omega_1^{1/2}, \ldots, \hat{\gamma}(p, J)\omega_J^{1/2}]' \tag{17}$$

$$A(p, J) = [\hat{\boldsymbol{\Gamma}}(p, 1)\omega_1^{1/2}, \ldots, \hat{\boldsymbol{\Gamma}}(p, J)\omega_J^{1/2}]' \tag{18}$$

In our simulations and empirical applications below, we focus on the case where the weights are equal *i.e.*, $\omega_j = 1/J$ where $j = 1, \ldots, J$.

# Forecasting

We use the Kalman filter to estimate volatility and produce forecasts.

Out-of-sample $h$ step-ahead forecasting:

$$\hat{\xi}_{T+h|T} = F^h \hat{\xi}_{T|T} + F^{(h-1)} \delta \sigma_\nu \eta_{T|T} \tag{19}$$

or

$$\hat{y}^*_{T+h|T} = H' \hat{\xi}_{T+h|T} \,. \tag{20}$$

where $\xi_t = [w_t, w_{t-1}, \ldots, w_{t-p+1}]'$.

From (19), we can see that the effect from leverage should decay as $h$ increases for a stationary process since it is weighted by $F^{(h-1)}$.

# Hypothesis Tests

We consider GMM-based LR-type statistics, based on the following moment-based objective function:

$$M_T(\theta) := g_T(\theta)' A_T \, g_T(\theta) \tag{21}$$

where $\theta := (\phi_1, \ldots, \phi_p, \sigma_y, \sigma_v, \delta)'$, $g_T(\theta)$ is $(p+3) \times 1$ vector of moments, defined as

$$g_T(\theta) = \begin{bmatrix} \hat{\mu} + 1.2704 - \log(\sigma_y^2) \\ \hat{\gamma}_{y^*}(0) + \hat{\gamma}_{y^*}(1) - (\pi^2/2) - (1-\phi_1)^{-1}\left(\sum_{j=2}^{p}\left[\phi_j(\hat{\gamma}_{y^*}(j-1) + \hat{\gamma}_{y^*}(j)\right] - \sigma_v^2\right) \\ \hat{\gamma}_{y^*}(p+1) - \phi_1\hat{\gamma}_{y^*}(p) + \cdots + \phi_p\hat{\gamma}_{y^*}(1) \\ \vdots \\ \hat{\gamma}_{y^*}(2p) - \phi_1\hat{\gamma}_{y^*}(2p-1) + \cdots + \phi_p\hat{\gamma}_{y^*}(p) \\ \delta - \frac{\sqrt{2\pi}\hat{\lambda}_y(1)}{\sigma_v\sigma_y^2}\exp\left(-\frac{1}{4}\tilde{\gamma}_p\right) \end{bmatrix} \tag{22}$$

and $A_T$ is an appropriate weighting matrix.

# Likelihood Ratio Test

Since the number of moment functions in (22) is equal to the number of parameters, we take $A_T = I_{(p+3)}$ so that

$$M^*_T(\theta) = g_T(\theta)' g_T(\theta) \tag{23}$$

Then, the LR-type statistic is given by:

$$LR_T = T[M^*_T(\hat{\theta}_0) - M^*_T(\hat{\theta})] \tag{24}$$

- $\hat{\theta}$: unrestricted estimator
- $\hat{\theta}_0$: constrained estimator under the null hypothesis
- $LR_T \sim \chi^2_r$: Under standard regularity conditions (which may not apply here); see Newey and West (1987), Newey and McFadden (1994), Dufour et al. (2017)

# Simulation-based finite-sample tests

Monte Carlo tests:

- Work with sample distribution of test statistic (see Dufour (2006))
  - Replace "theoretical" null distribution $F(x)$ of $LR_T$ with its simulation-based "estimate" $\hat{F}(x)$
  - Does not rely on existence of asymptotic distribution
- Use MMC and LMC procedures to deal with presence of nuisance parameters

Monte Carlo p-value is given by

$$\hat{p}_N[LR_T^{(0)}|\theta_0] = \frac{N + 1 - R_{LR}[LR_T^{(0)}; N]}{N + 1} \tag{25}$$

where $R_{LR}[LR_T^{(0)}; N] = \sum_{i=1}^{N} \mathbb{1}\{LR_T^{(0)} \geq LR_T^i\}$. Using proposition 4.1 of Dufour (2006) under the null hypothesis we have a valid test procedure.

# Monte Carlo Tests

- Maximized Monte Carlo (MMC) Test:
  - Requires searching over the parameter space $\bar{\Omega}_0$ consistent with null hypothesis
  - Another option is to search within a smaller consistent set of the parameter space $C_T$. For example, let $\hat{\theta}$ be the consistent point estimate of $\theta_0$. Then, we can define

$$C_T = \{\theta \in \Omega : \left\|\hat{\theta} - \theta_0\right\| < c\} \tag{26}$$

    where $c$ is a fixed positive constant that does not depend on $T$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^k$

- Local Monte Carlo ($LMC$) Test:
  - Define $C_T$ to be the singleton set i.e., $C_T = \hat{\theta}$

# Estimation: Comparison with Bayes & QML

Table 1: Comparison with competing estimators: Bias and RMSE

| | | | | Bias | | | | RMSE | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $\phi$ | $\sigma_y$ | $\sigma_v$ | $\delta$ | $\phi$ | $\sigma_y$ | $\sigma$ | $\delta$ |
| | | | | True Value | | | | | | | |
| $T$ | Estimators | RCT | NIV | 0.95 | 0.15 | 1 | -0.95 | 0.95 | 0.15 | 1 | -0.95 |
| 500 | Bayes | 31929.7 | 0 | -0.0568 | 0.0830 | 0.1343 | 0.6639 | 0.0617 | 0.1108 | 0.1530 | 0.6653 |
| | QML | 1028.8 | 0 | -0.0767 | 0.0176 | -0.8469 | 0.3481 | 0.2208 | 0.0810 | 1.2603 | 0.4151 |
| | CF-ARMA | 1.0 | 61 | -0.0142 | 0.0160 | -0.0146 | 0.1542 | 0.0396 | 0.0777 | 0.2822 | 0.3861 |
| | **W-ARMA** ($J = 10$) | 1.7 | 0 | -0.0137 | **0.0155** | **0.0155** | 0.1609 | **0.0268** | **0.0768** | **0.1231** | 0.3880 |
| | **W-ARMA** ($J = 100$) | 2.6 | 0 | **-0.0064** | **0.0155** | -0.0466 | **0.1521** | 0.0236 | **0.0768** | 0.1412 | **0.3818** |
| 2000 | Bayes | 218068.0 | 0 | -0.0155 | 0.0346 | -0.0151 | 0.1588 | 0.1070 | 0.0573 | 0.0621 | 0.1663 |
| | QML | 1025.5 | 0 | -0.0482 | **0.0022** | -0.4264 | 0.3273 | 0.1513 | 0.0363 | 0.8612 | 0.3855 |
| | CF-ARMA | 1.0 | 3 | -0.0031 | 0.0030 | -0.0238 | -0.0313 | 0.0188 | 0.0356 | 0.1678 | 0.1173 |
| | **W-ARMA** ($J = 10$) | 1.4 | 0 | -0.0038 | 0.0031 | **-0.0039** | **-0.0315** | **0.0105** | **0.0356** | **0.0617** | 0.1172 |
| | **W-ARMA** ($J = 100$) | 1.8 | 0 | **-0.0017** | 0.0031 | -0.0233 | -0.0322 | 0.0106 | **0.0356** | 0.0781 | **0.1158** |

Notes: We simulate 1000 samples from each model. W-ARMA ($J = 10, 100$) is the winsorized ARMA estimator based on OLS and $J$ is the winsorizing parameter. QML is the quasi-maximum likelihood estimator of Harvey and Shephard (1996). We used R package *stochvol* of Kastner (2016) for the Bayesian estimation based on Markov Chain Monte Carlo methods, where the posteriors are based on 50000 draws of the sampler, after discarding 50000 draws. RCT stands for the relative computational time w.r.t. the CF-ARMA estimator. The number of inadmissible values (NIV) of $\phi$ is also reported, these are out of 1000. Boldface font highlights the smallest bias and RMSE with no NIV. Boldface font also highlights the estimator, which has the best overall performance.

# Tests for no leverage - Empircial size

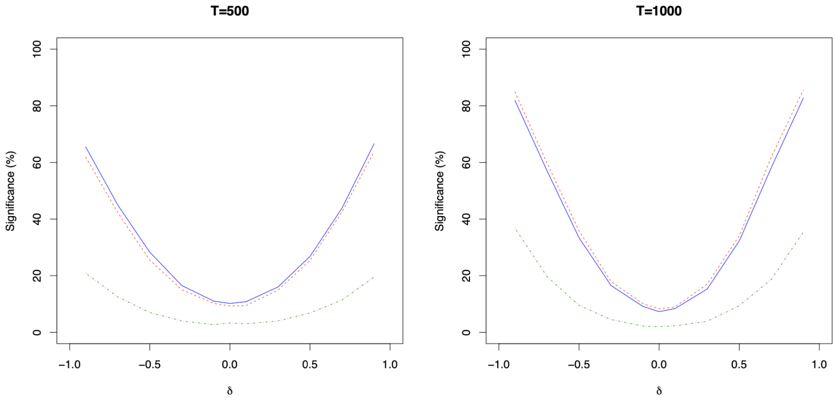
Table 2: Empirical size of tests for no leverage in SVL($p$) model

| | $H_0$: $\delta = 0$ vs. $H_1$: $\delta \neq 0$ | | | | | |
|---|---|---|---|---|---|---|
| $T$ | Asy | **LMC** | **MMC** | Asy | **LMC** | **MMC** |
| | $\phi = 0.90, \sigma_y = 0.10, \sigma_v = 0.75$ | | | $\phi = 0.75, \sigma_y = 0.10, \sigma_v = 1.00$ | | |
| 500 | 79.3 | 12.6 | 3.3 | 71.1 | 9.9 | 1.5 |
| 1000 | 82.8 | 12.8 | 2.0 | 72.8 | 10.1 | 1.4 |
| 2000 | 85.0 | 11.4 | 1.8 | 76.7 | 11.4 | 1.0 |
| 5000 | 89.0 | 10.2 | 1.0 | 77.9 | 10.8 | 1.4 |
| | $\phi = 0.99, \sigma_y = 0.10, \sigma_v = 0.25$ | | | $\phi = 0.95, \sigma_y = 0.10, \sigma_v = 0.50$ | | |
| 500 | 81.8 | 19.5 | 5.3 | 79.7 | 13.6 | 2.5 |
| 1000 | 88.7 | 17.1 | 5.2 | 83.3 | 13.4 | 2.3 |
| 2000 | 92.1 | 16.8 | 5.4 | 87.9 | 11.4 | 1.6 |
| 5000 | 93.2 | 13.3 | 4.8 | 91.2 | 10.9 | 1.4 |
| | $\phi_1 = 0.05, \phi_2 = 0.85, \sigma_y = 1.00, \sigma_v = 1.00$ | | | $\phi_1 = 0.05, \phi_2 = 0.70, \sigma_y = 1.00, \sigma_v = 1.00$ | | |
| 500 | 57.9 | 13.8 | 6.7 | 53.1 | 8.0 | 2.7 |
| 1000 | 62.8 | 10.7 | 3.9 | 54.3 | 6.3 | 1.7 |
| 2000 | 68.0 | 8.8 | 2.4 | 55.8 | 7.1 | 1.5 |
| 5000 | 77.0 | 8.4 | 1.9 | 57.1 | 6.8 | 2.5 |

Notes: Rejection frequencies are obtained using 1000 replications. Monte Carlo tests use $N = 99$ simulations.

# Tests for no leverage - Empirical power

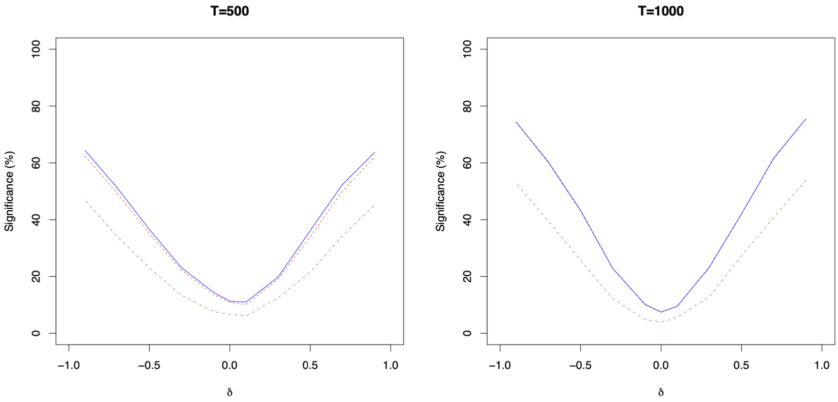
Figure 2: Power curves of test for no leverage in SVL(1) model




Notes: Here, $\phi = 0.90$, $\sigma_y = 0.10$, and $\sigma_v = 0.75$. Red is Asymptotic test (level corrected), blue is the Local Monte Carlo test, and green is the Maximized Monte Carlo test.

# Tests for no leverage - Empirical power

Figure 3: Power curves of test for no leverage in SVL(2) model




Notes: Here, $\phi_1 = 0.05$, $\phi_2 = 0.85$, $\sigma_y = 1.00$, and $\sigma_v = 1.00$. Red is Asymptotic test (level corrected), blue is the Local Monte Carlo test, and green is the Maximized Monte Carlo test.

# Forecasting

Table 3: MSE ratios and DM test for forecasting with leverage vs. no leverage

| | SV(1) vs. SVL(1) | | | | | |
|---|---|---|---|---|---|---|
| | $h=1$ | | $h=5$ | | $h=10$ | |
| $\delta$ | MSE ratio | DM Stat. | MSE ratio | DM Stat. | MSE ratio | DM Stat. |
| | | | $\phi = 0.90$, $\sigma_y = 0.10$, $\sigma_v = 0.75$ | | | |
| -0.9 | 1.14 | 7.86*** | 1.08 | 8.69*** | 1.05 | 8.75*** |
| -0.7 | 1.06 | 3.77*** | 1.04 | 5.71*** | 1.03 | 5.70*** |
| -0.5 | 1.02 | 1.75* | 1.02 | 3.64*** | 1.01 | 3.57*** |
| 0.5 | 1.02 | 2.67*** | 1.01 | 3.96*** | 1.01 | 4.54*** |
| 0.7 | 1.07 | 5.43*** | 1.04 | 6.91*** | 1.03 | 7.27*** |
| 0.9 | 1.15 | 7.94*** | 1.09 | 9.13*** | 1.05 | 9.11*** |

Notes: We simulate a process with $T = 2,010$ observations, estimate the model using the first $T_{est} = 2,000$, and forecast the last 10 observations. This process is repeated $B = 3,000$ times. The MSE ratio is computed by using the MSE of the SV($p$) model in the numerator and the MSE of the SVL($p$) model in the denominator, so that a value greater than 1 suggests that the model with leverage provides a better forecast. The reported values represent the Mean Squared Error (MSE) ratio for each model at different horizons: $h = 1$ (one day), $h = 5$ (one week), and $h = 10$ (two weeks). The DM statistic is computed as in equation (33). The significance level of the DM statistic is indicated using $(*)$ for 10% significance level, $(**)$ for 5% significance level, and $(***)$ for 1% significance level.

# Forecasting

Table 4: MSE ratios and DM test for forecasting with leverage vs. no leverage

| | SV(2) vs. SVL(2) | | | | | |
|---|---|---|---|---|---|---|
| | $h=1$ | | $h=5$ | | $h=10$ | |
| $\delta$ | MSE ratio | DM Stat. | MSE ratio | DM Stat. | MSE ratio | DM Stat. |
| | $\phi_1 = 0.05$, $\phi_2 = 0.85$, $\sigma_y = 1.00$, $\sigma_v = 1.00$ | | | | | |
| -0.9 | 1.14 | 7.87*** | 1.03 | 7.12*** | 1.01 | 7.07*** |
| -0.7 | 1.08 | 6.59*** | 1.02 | 5.43*** | 1.01 | 5.50*** |
| -0.5 | 1.04 | 4.86*** | 1.01 | 3.94*** | 1.00 | 4.00*** |
| 0.5 | 1.04 | 5.08*** | 1.01 | 4.11*** | 1.00 | 4.37*** |
| 0.7 | 1.08 | 6.59*** | 1.01 | 5.96*** | 1.01 | 6.21*** |
| 0.9 | 1.14 | 7.97*** | 1.03 | 7.86*** | 1.01 | 7.95*** |

Notes: We simulate a process with $T = 2,010$ observations, estimate the model using the first $T_{est} = 2,000$, and forecast the last 10 observations. This process is repeated $B = 3,000$ times. The MSE ratio is computed by using the MSE of the SV($p$) model in the numerator and the MSE of the SVL($p$) model in the denominator, so that a value greater than 1 suggests that the model with leverage provides a better forecast. The reported values represent the Mean Squared Error (MSE) ratio for each model at different horizons: $h = 1$ (one day), $h = 5$ (one week), and $h = 10$ (two weeks). The DM statistic is computed as in equation (33). The significance level of the DM statistic is indicated using $(*)$ for 10% significance level, $(**)$ for 5% significance level, and $(***)$ for 1% significance level.

# Empirical estimation: Three stock indices

Table 5: Empirical W-ARMA estimates of SVL($p$) models

| | SVL(1) | | | | SVL(2) | | | | | SVL(3) | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\hat{\phi}$ | $\hat{\sigma}_y$ | $\hat{\sigma}_\nu$ | $\hat{\delta}$ | $\hat{\phi}_1$ | $\hat{\phi}_2$ | $\hat{\sigma}_y$ | $\hat{\sigma}_\nu$ | $\hat{\delta}$ | $\hat{\phi}_1$ | $\hat{\phi}_2$ | $\hat{\phi}_3$ | $\hat{\sigma}_y$ | $\hat{\sigma}_\nu$ | $\hat{\delta}$ |
| S&P 500 | | | | | | | | | | | | | | | |
| est. | 0.972 | 0.850 | 0.294 | **-0.774** | 0.427 | 0.549 | 0.850 | 0.719 | **-0.024** | 0.141 | 0.355 | 0.477 | 0.850 | 0.639 | **-0.151** |
| SE | (0.001) | (0.055) | (0.011) | (0.115) | (0.033) | (0.031) | (0.131) | (0.056) | (0.498) | (0.006) | (0.086) | (0.009) | (0.109) | (0.075) | (0.171) |
| DOWJ | | | | | | | | | | | | | | | |
| est. | 0.970 | 0.811 | 0.294 | **-0.749** | 0.513 | 0.459 | 0.811 | 0.689 | **-0.035** | 0.300 | 0.235 | 0.434 | 0.811 | 0.628 | **-0.154** |
| SE | (0.001) | (0.049) | (0.011) | (0.128) | (0.074) | (0.072) | (0.111) | (0.061) | (0.492) | (0.076) | (0.084) | (0.027) | (0.093) | (0.072) | (0.193) |
| NASDQ | | | | | | | | | | | | | | | |
| est. | 0.985 | 1.135 | 0.209 | **-0.998** | 0.424 | 0.563 | 1.135 | 0.621 | **-0.008** | 0.234 | 0.448 | 0.302 | 1.135 | 0.598 | **-0.046** |
| SE | (0.001) | (0.099) | (0.005) | (0.001) | (0.031) | (0.031) | (0.25) | (0.070) | (0.506) | (0.044) | (0.047) | (0.080) | (0.217) | (0.076) | (0.487) |

Notes: Sample for each index is from 2000-Jan-04 to 2023-May-31 ($T = 5,889$). Estimates are obtained using WARMA estimator given in (16) with $J = 250$.

# Empirical test for leverage

Table 6: Empirical asymptotic and finite-sample tests for the presence of leverage in SVL($p$) models

| $H_0$: $\delta = 0$ vs. $H_1$: $\delta \neq 0$ | SVL(1) | | | SVL(2) | | | SVL(3) | | |
|---|---|---|---|---|---|---|---|---|---|
| | Asymptotic | **LMC** | **MMC** | Asymptotic | **LMC** | **MMC** | Asymptotic | **LMC** | **MMC** |
| S&P 500 | 0.00 | 0.00 | 0.01 | 0.07 | 0.83 | 0.95 | 0.00 | 0.32 | 0.70 |
| Dow Jones | 0.00 | 0.00 | 0.01 | 0.01 | 0.84 | 0.94 | 0.00 | 0.37 | 0.46 |
| NASDAQ | 0.00 | 0.00 | 0.01 | 0.56 | 0.87 | 0.92 | 0.00 | 0.70 | 0.89 |

Notes: Sample for each index is from 2000-Jan-04 to 2023-May-31 ($T = 5,889$). The reported values are p-values for test procedure when testing for leverage (*i.e.* $H_0 : \delta = 0$ vs. $H_1 : \delta \neq 0$). When estimating the constrained and unconstrained models we used WARMA estimator given in (16) with $J = 250$. We use $N = 999$ Monte Carlo simulations to simulate the null distribution for LMC and MMC tests.

# Empirical Forecast

Table 7: Empirical forecast with $T = 1,000$ rolling estimation window

| | S&P 500 | | | Dow Jones | | | NASDAQ | | |
|---|---|---|---|---|---|---|---|---|---|
| Models | $h=1$ | $h=5$ | $h=10$ | $h=1$ | $h=5$ | $h=10$ | $h=1$ | $h=5$ | $h=10$ |
| EGARCH(1,1) | 7.971 | 40.683 | 82.891 | **8.006** | **40.792** | 83.014 | 7.674 | 38.927 | **78.895** |
| EGARCH(2,2) | 7.878 | 40.272 | 82.137 | **7.941** | **40.486** | 82.466 | 8.109 | 40.711 | **82.047** |
| EGARCH(3,3) | 7.792 | 39.916 | 81.499 | **11.809** | **59.887** | **110.934** | 9.855 | 53.878 | **161.959** |
| GJR(1,1) | 8.059 | 41.551 | 85.553 | 8.051 | 41.430 | 85.102 | 7.723 | 39.313 | **79.968** |
| GJR(2,2) | 8.034 | 41.432 | 85.409 | 8.030 | **41.267** | 84.842 | 7.702 | 39.206 | **79.821** |
| GJR(3,3) | 8.024 | 41.372 | 85.335 | 8.036 | **41.280** | 84.912 | 7.690 | 39.189 | **79.825** |
| SV(1) | 6.261 | **31.721** | **64.657** | **6.191** | **31.379** | **63.867** | 6.088 | **30.669** | **62.200** |
| SV(2) | 6.257 | 32.863 | 67.359 | **6.183** | **32.433** | **66.462** | 6.027 | 31.370 | **63.962** |
| SV(3) | **6.193** | 33.077 | 67.616 | **6.092** | **32.704** | **66.798** | 5.987 | 31.607 | **64.239** |
| SVL(1) | **6.148** | **31.526** | **64.508** | **6.102** | **31.128** | **63.579** | **5.949** | **30.323** | **61.786** |
| SVL(2) | **6.174** | 32.690 | 67.183 | **6.118** | **32.262** | **66.283** | **5.927** | 31.186 | **63.775** |
| SVL(3) | **6.097** | 32.915 | 67.451 | **6.007** | **32.533** | **66.623** | **5.875** | 31.432 | **64.063** |

Notes: ARCH & GARCH models also considered but not shown here due to space. The reported values represent the MSE for each model at different horizons. The values in bold indicate that the model is part of the Model Confidence Set (MCS). The MCS is determined using a 5% significance level. The values in **bold red** indicate the **models in the MCS with the lowest MSE**. Estimates are obtained using WARMA estimator given in (16) with $J = 250$. Out-of-sample forecasting is performed using a rolling window scheme of size $T_{est} = 1,000$.

# Conclusion

## In this paper

- Propose moment-based simple closed-form estimator for SVL($p$)
  - Computationally efficient
  - W-ARMA estimators further improve the stability particularly in the presence of outliers or small samples
- Finite-sample Monte Carlo tests for no leverage $H_0 : \delta = 0$
  - Control size and power of LR-type tests
- Empirical app. with daily returns of S&P 500, DOWJ, and NASDAQ
  - Find evidence of leverage when using SVL(1)
  - Highlight significance of leverage in volatility forecasting

Thank you!

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# Estimation: $\mathbf{\Gamma}(p,j)$

$\mathbf{\Gamma}(p,j)$ is the following $p \times p$ matrix:

$$\mathbf{\Gamma}(p,\, j) := \begin{bmatrix} \gamma_{y^*}(j+p-1) & \gamma_{y^*}(j+p-2) & \cdots & \gamma_{y^*}(j) \\ \gamma_{y^*}(j+p) & \gamma_{y^*}(j+p-1) & \cdots & \gamma_{y^*}(j+1) \\ \vdots & \vdots & & \vdots \\ \gamma_{y^*}(j+2p-2) & \gamma_{y^*}(j+2p-3) & \cdots & \gamma_{y^*}(j+p-1) \end{bmatrix}, \tag{27}$$

where $p$ is the SV order, $\gamma_{y^*}(k) = \mathrm{cov}(y_t^*, y_{t-k}^*)$, with $y_t^* = [\log(y_t^2) - \mu]$ and $\mu := \mathbb{E}[\log(y_t^2)]$.

# Kalman Filter: Estimate volatility

We use Kalman filter to get estimate of unobserved volatility process & perform forecasts

$$y_t^* = H'\xi_t + \epsilon_t \tag{28}$$

$$\xi_{t+1} = F\xi_t + \sigma_v u_{t+1}\,, \tag{29}$$

where $H' = [1, 0, \ldots, 0]$ is a $1 \times p$ vector, $u$ and

$$u_{t+1} = \delta\eta_t + (1-\delta^2)^{1/2}\zeta_{t+1}, \qquad E[u_t u_t'] = E[z_t z_t'] = E[\tilde{\nu}_t \tilde{\nu}_t'] = \Omega_1,$$

$$\xi_t = \begin{bmatrix} w_t \\ w_{t-1} \\ w_{t-2} \\ \vdots \\ w_{t-p+1} \end{bmatrix}, \; F = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \; \eta_t = \begin{bmatrix} z_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; \zeta_t = \begin{bmatrix} \tilde{\nu}_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; u_{t+1} = \begin{bmatrix} \nu_{t+1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and where $F$ and $\Omega_1$ are $p \times p$ matrices, and $\xi_t$, $u_t$, $\eta_t$, $\zeta_t$ are $p \times 1$ vectors.

# Maximized Monte Carlo Test

Monte Carlo p-value is given by

$$\hat{p}_N[LR_T^{(0)}|\theta_0] = \frac{N + 1 - R_{LR}[LR_T^{(0)}; N]}{N + 1} \tag{30}$$

where $R_{LR}[LR_T^{(0)}; N] = \sum_{i=1}^{N} \mathbb{1}\{LR_T^{(0)} \geq LR_T^{i}\}$. Using proposition 4.1 of Dufour (2006) under the null hypothesis we have

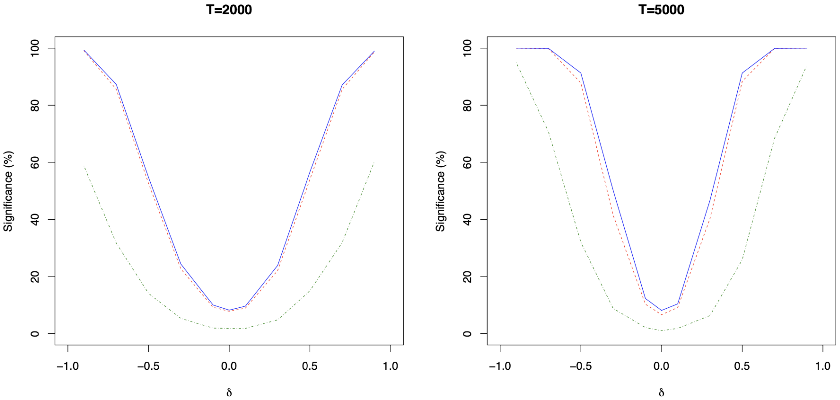
$$Pr\left[\sup\left\{\hat{p}_N[LR_T^{(0)}|\theta_0] : \theta_0 \in \bar{\Omega}_0\right\} \leq \alpha\right] \leq \alpha$$

A valid test procedure. To search over the parameter space $\bar{\Omega}_0$ we can use:

- Generalized Simulated Annealing
- Genetic Algorithms
- Particle Swarm

# Tests for no leverage - Empirical power

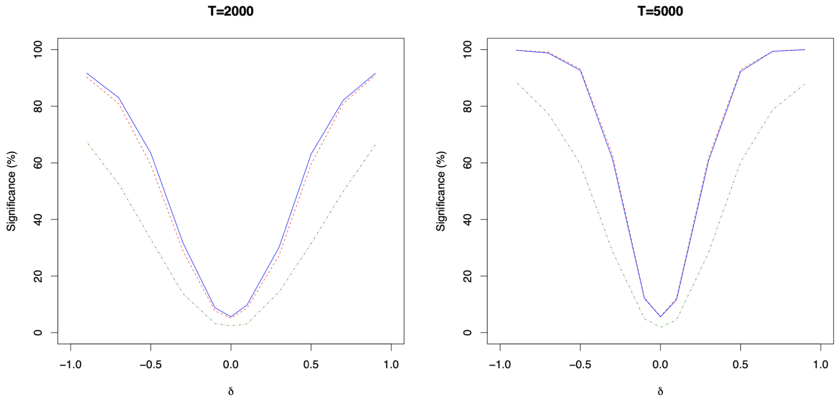
Figure 4: Power curves of test for no leverage in SVL(1) model




Notes: Here, $\phi = 0.90$, $\sigma_y = 0.10$, and $\sigma_v = 0.75$. Red is Asymptotic test (level corrected), blue is the Local Monte Carlo test, and green is the Maximized Monte Carlo test.

# Tests for no leverage - Empirical power

Figure 5: Power curves of test for no leverage in SVL(2) model




Notes: Here, $\phi_1 = 0.05$, $\phi_2 = 0.85$, $\sigma_y = 1.00$, and $\sigma_v = 1.00$. Red is Asymptotic test (level corrected), blue is the Local Monte Carlo test, and green is the Maximized Monte Carlo test.

# Forecasting

Specifically, the *MSE* of each model is computed as,

$$MSE_m = \frac{1}{B}\sum_{i=1}^{B}\left(\sum_{j=1}^{h}\left[\log(y_{t+j}^2) - \log(\hat{y}_{m,t+j|t}^2)\right]^2\right), \tag{31}$$

where $m = \{SV, SVL\}$ and $B$ is the number of simulations. The ratio is then computed as $MSE$ ratio $= MSE_{SV}/MSE_{SVL}$ and hence, a value greater than 1 suggests the model with leverage performs better.

# Forecasting

We use the DM test of Diebold and Mariano (2002) as a means to determine the statistical significance of the difference in the forecast performance of a model with leverage and a model with no leverage.

$$d_i = \sum_{j=1}^{h} \left[\log(y_{t+j}^2) - \log(\hat{y}_{SV,t+j|t}^2)\right]^2 - \sum_{j=1}^{h} \left[\log(y_{t+j}^2) - \log(\hat{y}_{SVL,t+j|t}^2)\right]^2 \quad (32)$$

From here, the DM statistic is computed as

$$DM = \frac{\bar{d}}{\sqrt{(\gamma_d(0) + 2\sum_{k=1}^{B^{1/3}} \gamma_d(k))/B}} \quad (33)$$

where $\bar{d} = \frac{1}{B}\sum_{i=1}^{B} d_i$ and $\gamma_d(k)$ is the sample auto-covariance of $d_i$ at lag $k$. As described in Diebold and Mariano (2002) this test statistic has a standard normal distribution under the null hypothesis of no statistical difference is the forecast error.

# Forecasting

Table 8: MSE ratios and DM test for forecasting with leverage vs. no leverage

| | SV(1) vs. SVL(1) | | | | | |
|---|---|---|---|---|---|---|
| | $h = 1$ | | $h = 5$ | | $h = 10$ | |
| $\delta$ | MSE ratio | DM Stat. | MSE ratio | DM Stat. | MSE ratio | DM Stat. |
| | $\phi = 0.75$, $\sigma_y = 0.10$, $\sigma_v = 1.00$ | | | | | |
| -0.9 | 1.19 | 8.13*** | 1.06 | 8.56*** | 1.03 | 8.57*** |
| -0.7 | 1.10 | 5.91*** | 1.04 | 7.32*** | 1.02 | 7.35*** |
| -0.5 | 1.04 | 3.42*** | 1.02 | 5.51*** | 1.01 | 5.54*** |
| 0.5 | 1.03 | 1.92* | 1.01 | 1.82* | 1.00 | 1.99** |
| 0.7 | 1.08 | 4.65*** | 1.03 | 5.62*** | 1.02 | 5.75*** |
| 0.9 | 1.19 | 7.92*** | 1.06 | 8.69*** | 1.03 | 8.70*** |

Notes: We simulate a process with $T = 2,010$ observations, estimate the model using the first $T_{est} = 2,000$, and forecast the last 10 observations. This process is repeated $B = 3,000$ times. The MSE ratio is computed by using the MSE of the SV($p$) model in the numerator and the MSE of the SVL($p$) model in the denominator, so that a value greater than 1 suggests that the model with leverage provides a better forecast. The reported values represent the Mean Squared Error (MSE) ratio for each model at different horizons: $h = 1$ (one day), $h = 5$ (one week), and $h = 10$ (two weeks). The DM statistic is computed as in equation (33). The significance level of the DM statistic is indicated using $(*)$ for 10% significance level, $(**)$ for 5% significance level, and $(***)$ for 1% significance level.

# Forecasting

Table 9: MSE ratios and DM test for forecasting with leverage vs. no leverage

| | SV(2) vs. SVL(2) | | | | | |
|---|---|---|---|---|---|---|
| | $h = 1$ | | $h = 5$ | | $h = 10$ | |
| $\delta$ | MSE ratio | DM Stat. | MSE ratio | DM Stat. | MSE ratio | DM Stat. |
| | $\phi_1 = 0.05$, $\phi_2 = 0.70$, $\sigma_y = 1.00$, $\sigma_v = 1.00$ | | | | | |
| -0.9 | 1.17 | 7.68*** | 1.03 | 7.78*** | 1.01 | 7.78*** |
| -0.7 | 1.09 | 5.94*** | 1.02 | 6.10*** | 1.01 | 6.09*** |
| -0.5 | 1.04 | 3.73*** | 1.01 | 3.98*** | 1.00 | 3.97*** |
| 0.5 | 1.04 | 4.37*** | 1.01 | 4.39*** | 1.00 | 4.44*** |
| 0.7 | 1.09 | 6.43*** | 1.02 | 6.44*** | 1.01 | 6.45*** |
| 0.9 | 1.18 | 7.98*** | 1.03 | 8.12*** | 1.01 | 8.12*** |

Notes: We simulate a process with $T = 2,010$ observations, estimate the model using the first $T_{est} = 2,000$, and forecast the last 10 observations. This process is repeated $B = 3,000$ times. The MSE ratio is computed by using the MSE of the SV($p$) model in the numerator and the MSE of the SVL($p$) model in the denominator, so that a value greater than 1 suggests that the model with leverage provides a better forecast. The reported values represent the Mean Squared Error (MSE) ratio for each model at different horizons: $h = 1$ (one day), $h = 5$ (one week), and $h = 10$ (two weeks). The DM statistic is computed as in equation (33). The significance level of the DM statistic is indicated using $(*)$ for 10% significance level, $(**)$ for 5% significance level, and $(***)$ for 1% significance level.